

# Constraints on Natural MNS Parameters from $|U_{e3}|$

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## Abstract

The MNS matrix structure emerging as a result of recent neutrino measurements strongly suggests two large mixing angles (solar and atmospheric) and one small angle ( $|U_{e3}| \ll 1$ ). Especially when combined with the neutrino mass hierarchy, these values turn out to impose rather stringent constraints on possible flavor models connecting the three active fermion generations. Specifically, we show that an extremely small value of  $|U_{e3}|$  would require fine tuning of Majorana mass matrix parameters, particularly in the context of seesaw models.

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## I. INTRODUCTION

Recent progress in experimental neutrino physics has been nothing short of breathtaking. At the time of this writing, only five years have elapsed since Super-Kamiokande’s observation [1] of the atmospheric neutrino deficit. Just last year the SNO experiment showed [2] through detection of neutral current events that the solar neutrino deficit, as observed by decades of steadily improving charged current experiments (Homestake [3], Kamiokande [4], SAGE [5], GALLEX [6], GNO [7], Super-Kamiokande [8], and SNO itself [2]) is truly a deficit due to solar  $\nu_e$ ’s oscillating into other active flavors. In the past year the K2K experiment [9] confirmed nicely the atmospheric mixing parameters measured by Ref. [1] using long-baseline accelerator techniques, while the KamLAND reactor experiment [10] tightly constrained the mixing parameter space previously probed only through solar neutrino measurements, and the WMAP satellite experiment [11] used cosmological constraints to bound the neutrino mass sum to lie below 0.71 eV (at 95% CL).

This list does not do justice to the large number of completed neutrino experiments that provide important exclusionary bounds, nor the panoply of new experiments underway or in the planning stages. Our intent is merely to provide a snapshot of a rapidly maturing field, and to point out that, even now, we possess a reasonably good picture of neutrino masses and mixing as encapsulated by the Maki-Nakagawa-Sakata (MNS) matrix. According to a recent three-flavor analysis [12], the parameters with  $2\sigma$  uncertainties are given by

$$|\Delta m_{32}^2| = (1.8 - 3.3) \times 10^{-3} \text{eV}^2, \quad (1.1)$$

$$\sin^2 \theta_{23} = 0.36 - 0.67, \quad (1.2)$$

$$\Delta m_{21}^2 = (6.0 - 8.4) \times 10^{-5} \text{eV}^2, \quad (1.3)$$

$$\sin^2 \theta_{12} = 0.25 - 0.36, \quad (1.4)$$

$$\sin^2 \theta_{13} \leq 0.035, \quad (1.5)$$

where the magnitude symbols on  $\Delta m_{32}^2$  serve as a reminder that either the normal or inverted hierarchy is still allowed. The important point here is not the precise values of the observables (which seem to improve monthly in response to new data regularly being released), but rather the picture painted by their magnitudes: All neutrino masses are small, their mass differences are smaller still and hierarchical, and precisely two of the three Euler angles in the MNS matrix are large (“bimaximal”), while the third is quite small (a result of including the

CHOOZ [13] and Palo Verde [14] exclusionary data in the global analysis). This structure differs radically from that found in the quark sector, where the CKM matrix possesses three small and hierarchically decreasing Euler angles (*i.e.*,  $1 \gg |V_{us}| \gg |V_{cb}| \gg |V_{ub}|$ ).

It is natural to suppose that the existence of three complete flavors (generations) of spin- $\frac{1}{2}$  fermions, with members possessing the same quantum numbers under all standard model gauge groups and differing only in mass, suggests some sort of flavor symmetry. This is, after all, how the periodic table of the elements and the quark structure of hadrons came to be understood. In the case of an ostensible flavor symmetry, however, one must incorporate a number of disparate properties: Masses extend from  $m_t \simeq 174$  GeV, near the weak scale, through those of a number of light quarks and charged leptons in the MeV to GeV range, down to the neutrinos, with masses (perhaps) in the meV range. A flavor symmetry must also accommodate the aforementioned differences between the CKM and MNS matrices. Finally, if the flavor symmetry leaves a signature detectable at weak scales (such as in the context of supersymmetry), it must avoid creating unacceptably large flavor-changing neutral currents.

In the case of the neutrinos, the seesaw mechanism provides a natural machinery for obtaining tiny mass eigenvalues, and in the following we employ the mechanism in its simplest form. Of course, the seesaw also requires neutrinos with a Majorana character, and the presence of Majorana in addition to Yukawa (Dirac) mass matrices both complicates and enriches the possible theory parameter space. For our current purposes, however, we are interested foremost in the general nature of the (derived) left-handed Majorana mass matrix as well as those of the Yukawa and right-handed Majorana matrices that generate it, but less so in their fundamental origins. Although the construction of a model to describe the origin of neutrino couplings is theoretically important, one must consider the low-energy phenomenological consequences of the model first in order to understand its limitations. The emerging precision of findings such as those in Eqs. (1.1)–(1.5), based on data only a few months old, suggests that we are rapidly approaching a “tipping point” at which exceptionally few Yukawa and Majorana matrix textures, and consequently a rarified set of possible flavor symmetries, will remain viable.

The purpose of this paper is to explore constraints on natural textures that simultaneously give rise to bimaximal atmospheric ( $\theta_{23}$ ) and solar ( $\theta_{12}$ ) angles, while maintaining a small  $\theta_{13}$  and a mass splitting hierarchy  $\Delta m_{21}^2/|\Delta m_{32}^2| \ll 1$ . We show, using only naturalness criteria

such as the assumption that the small value of  $\theta_{13}$  is not due to a fortuitous cancellation between charged lepton Yukawa and neutrino Majorana parameters, that there is an effective lower bound on the natural size of  $\theta_{13}$  values arising from flavor models that predict the elements of the Majorana and Yukawa matrices. In particular,  $\theta_{13}$  cannot be much smaller than its currently measured bound, unless one resorts to fine tuning in either or both of the Yukawa and Majorana matrices. A recent paper [15], in the same spirit of studying mass matrix textures, also finds that  $\theta_{13}$  should lie close to its current bound; however, Ref. [15] places emphasis on texture zeroes instead of (as here) the naturalness constraints of allowing matrix elements to be as large as possible.

We note that the prediction that  $\theta_{13}$  is not expected to be excessively small has been made in several previous works [16, 17, 18, 19], in the context of various models and parametrizations. Moreover, Ref. [20] studies this issue by examining classes of models appearing in the literature and finds a similar effect. In this paper we derive and explain a simple underlying reason for this recurring result.

The paper is organized as follows: In Sec. II we establish the conventions defining the relevant matrices. In Sec. III we lay out the naturalness criteria used to study Yukawa and Majorana textures and see how they are realized in a sample model. Section IV presents a constructive method of building the left-handed neutrino Majorana matrix that satisfies all existing data, at which point we observe how little effect a sufficiently small  $\theta_{13}$  would have on the textures. Sections V and VI explore the consequences of these constraints for the textures of the heavy right-handed neutrinos entering the seesaw mechanism in the normal and inverted hierarchies, respectively. Section VII concludes.

## II. CONVENTIONS AND NOTATIONS

The conventions defining the charged lepton Yukawa matrix  $Y_L$ , the neutrino Yukawa matrix  $M_{LR}$ , and the neutrino Majorana mass matrices  $M_{LL}$  and  $M_{RR}$  are established simply by presenting the relevant Lagrangian terms. What follows is of course well known, but is included in order to leave no ambiguities in the notation used in the rest of the paper. Beginning with the neutrino terms,

$$\mathcal{L}_\nu = \bar{\nu}_L i \not{\partial} \nu_L + \bar{\nu}_R i \not{\partial} \nu_R - \frac{1}{2} (\bar{\nu}_L^c M_{LL} \nu_L + \text{H.c.}) - \frac{1}{2} (\bar{\nu}_R^c M_{RR} \nu_R + \text{H.c.}) - (\bar{\nu}_L M_{LR} \nu_R + \text{H.c.}), \quad (2.1)$$

where  $\psi^c \equiv C\bar{\psi}^T$  is the field obtained via the charge conjugation operation  $C$ , H.c. = Hermitian conjugate, and the fields  $\nu_{L,R}$  stand for spinors containing all three neutrino generations, one first notes that the presence of a transposed (rather than Hermitian-conjugated) field in the ket of Majorana terms constrains  $M_{LL}$  and  $M_{RR}$  both to be (complex) symmetric matrices. In the most minimal seesaw mechanism,  $M_{LL}$  does not enter as a fundamental field in the Lagrangian, but rather arises as an effective operator:

$$M_{LL}^{\text{eff}} = -M_{LR}^* M_{RR}^{-1} M_{LR}^\dagger . \quad (2.2)$$

As discussed in the Introduction, in this work we begin by treating the effective  $M_{LL}$  precisely as one would any fundamental Lagrangian parameter.

The Lagrangian in the lepton sector is then defined by:

$$\begin{aligned} \mathcal{L} = & \bar{\nu}_L i \not{\partial} \nu_L + \bar{e}_L i \not{\partial} e_L + \bar{e}_R i \not{\partial} e_R - \frac{1}{2} (\bar{\nu}_L^c M_{LL} \nu_L + \text{H.c.}) - \frac{v}{\sqrt{2}} (\bar{e}_L Y_L e_R + \text{H.c.}) \\ & + \frac{g_W}{\sqrt{2}} (\bar{e}_L \mathcal{W}^- \nu_L + \text{H.c.}) , \end{aligned} \quad (2.3)$$

where  $g_W$  is the usual electroweak gauge coupling and  $v \simeq 246$  GeV. We diagonalize  $Y_L$  in the usual way, with a biunitary transformation given by

$$Y_L^0 = U_L^\dagger Y_L U_R , \quad (2.4)$$

where  $Y_L^0$  is diagonal. That is, the mass basis is related to the weak interaction basis by  $e_L^0 = U_L^\dagger e_L$ ,  $e_R^0 = U_R^\dagger e_R$ , and  $U_L$  is obtained by diagonalizing  $Y_L Y_L^\dagger$ :

$$\begin{aligned} Y_L^0 Y_L^{0\dagger} &= U_L^\dagger Y_L U_R U_R^\dagger Y_L^\dagger U_L \\ &= U_L^\dagger Y_L Y_L^\dagger U_L , \end{aligned} \quad (2.5)$$

while  $U_R$  may be obtained by diagonalizing  $Y_L^\dagger Y_L$ .

Since the most general Majorana mass matrix  $M_{LL}$  is complex symmetric rather than Hermitian, the usual procedure of diagonalization via a unitary matrix composed of its eigenvectors is no longer entirely valid. Nevertheless, one can prove [21] that for any such  $M_{LL}$  there exists a unitary matrix  $W$  such that

$$M_{LL}^0 = W^T M_{LL} W \quad (2.6)$$

is diagonal (in mathematical jargon, “is reduced to normal form”), and  $\nu_L^0 = W^\dagger \nu_L$ . The kinetic and mass terms are unaffected by these transformations, and the interaction term

becomes

$$\mathcal{L}_{\text{int}} = +\frac{g_W}{\sqrt{2}} \left( \bar{e}_L^0 \mathcal{W}^- U_L^\dagger W \nu_L^0 + \text{H.c.} \right), \quad (2.7)$$

from which one defines the MNS matrix as

$$U_{\text{MNS}} = U_L^\dagger W. \quad (2.8)$$

The full MNS matrix provides the linear combination of freely propagating neutrino mass eigenstates that interacts at a weak vertex with a particular charged lepton mass eigenstate. For example, Eqs. (2.7) and (2.8) show that an electron is produced from the “isolated” neutrino mass eigenstate  $\nu_3^0$  with an amplitude given by (dropping the MNS subscript) the now-famous matrix element  $U_{e3}$ .

Although the construction of  $U_{\text{MNS}}$  closely resembles that of the CKM matrix, the Majorana nature of the neutrinos permits two additional observable phases. Using abbreviations  $\sin \theta_{ij} \equiv s_{ij}$  and  $\cos \theta_{ij} \equiv c_{ij}$ , one convenient parametrization [22] for  $U_{\text{MNS}}$  reads

$$U_{\text{MNS}} = \begin{pmatrix} c_{13}c_{12} e^{i\beta_1} & c_{13}s_{12} e^{i\beta_2} & s_{13} e^{-i\delta} \\ (-s_{12}c_{23} - s_{13}c_{12}s_{23} e^{i\delta}) e^{i\beta_1} & (+c_{12}c_{23} - s_{13}s_{12}s_{23} e^{i\delta}) e^{i\beta_2} & c_{13}s_{23} \\ (+s_{12}s_{23} - s_{13}c_{12}c_{23} e^{i\delta}) e^{i\beta_1} & (-c_{12}s_{23} - s_{13}s_{12}c_{23} e^{i\delta}) e^{i\beta_2} & c_{13}c_{23} \end{pmatrix}. \quad (2.9)$$

The two CP-violating parameters  $\beta_{1,2}$  represent relative phases between the neutrino mass eigenstates and can be removed from  $U_{\text{MNS}}$  in Eqs. (2.7)–(2.8); they do not contribute to neutrino oscillation measurements [23], leaving  $\delta$  as the sole CP-violating parameter discernable in such data. However, the Majorana mass term is not invariant under such a redefinition of phases, meaning that only experiments sensitive to the Majorana nature of neutrinos, such as neutrinoless double beta decay, can probe  $\beta_{1,2}$ . Notably, the conventions are chosen so that the phase  $\delta$  multiplies the smallest Euler angle  $\theta_{13}$ , thus minimizing its impact in the texture.

### III. CONTRIBUTIONS TO THE MNS MATRIX

#### A. General Features

As is clear from Eq. (2.8), the MNS matrix receives contributions from both the unitary matrix  $U_L$  that diagonalizes the charged lepton Yukawa matrix combination  $Y_L Y_L^\dagger$  and the unitary matrix  $W$  that puts the neutrino Majorana mass matrix  $M_{LL}$  into normal form. In

general, both matrices can contribute equally to the observed mixing angles. However, as we argue in this section, such a democratic structure leads to either phenomenological or theoretical problems if one supposes that a natural flavor symmetry underlies these textures. To be specific, it does not appear possible to satisfy simultaneously the following criteria:

1. The Yukawa matrices of all light fermions—at least the charged ones—share a common origin in some flavor symmetry.
2. Hierarchies of charged fermion masses and quark mixing angles have a common origin, arising through the presence of small flavor symmetry-breaking parameters occupying certain elements of the Yukawa matrices.
3. There are no excessive fine-tuning cancellations between elements of  $U_L$  and  $W$  in forming  $U_{\text{MNS}}$  (or the unitary matrices that diagonalize the two left-handed quark Yukawa matrices and that comprise the CKM matrix).
4. The value of  $|U_{e3}|$  is very small (in a sense made explicit below).

Point (1) is the assertion that the texture of the Yukawa matrix  $Y_L$  is the same as those of  $Y_U$  and  $Y_D$  in the quark sector: Corresponding elements of  $Y_L$ ,  $Y_U$ , and  $Y_D$  differ numerically, of course, but have the same order of magnitude. Such a pattern occurs very naturally in grand unified theories. If this assumption fails, then the basic assumption that a flavor symmetry connects all fermions in a given generation falls into doubt. Point (2) asserts that the observed hierarchies, such as  $m_e \ll m_\mu \ll m_\tau$  and  $1 \gg |V_{us}| \gg |V_{cb}| \gg |V_{ub}|$ , arise from powers of the *same* small parameters (which are as few in number as possible); this of course is the usual minimality assumption. Point (3) is a naturalness assumption, that no small numbers should arise unless placed into the theory by hand.

Now, the fact that the each CKM angle is small, or equivalently that the CKM matrix is only perturbatively different from the identity matrix  $\mathbb{1}$ , implies that [by Point (3)] the separate unitary matrices that diagonalize the quark Yukawa matrices  $Y_U Y_U^\dagger$  and  $Y_D Y_D^\dagger$  are each only perturbatively different from being diagonal in the same basis. This, in turn, implies that  $Y_U Y_U^\dagger$  and  $Y_D Y_D^\dagger$  are each only perturbatively different from being diagonal themselves [32]. By the same reasoning, it would also be unnatural for a Yukawa combination such as  $Y_U Y_U^\dagger$  to be nearly diagonal and yet for  $Y_U$  itself to have large off-diagonal elements.

The notion that the quark Yukawa matrices themselves differ only perturbatively from being diagonal matrices is a feature common to virtually all flavor models. Indeed, perhaps the prettiest prediction of flavor models (and its oldest [25]) is the relation  $V_{us} \approx \sqrt{m_d/m_s}$ , which is obtained by supposing that the upper  $2 \times 2$  block of  $Y_D$  assumes the basic form

$$Y_D|_{2 \times 2} = \begin{pmatrix} 0 & O(\alpha) \\ O(\alpha) & 1 \end{pmatrix}, \quad (3.1)$$

where  $\alpha = O(V_{us}) = O(0.1)$  is a small flavor symmetry-breaking parameter, leading to a ratio of  $m_d$  to  $m_s$  eigenvalues in the phenomenologically correct proportion  $\alpha^2:1$ . The relation of CKM and quark mass parameters is, of course, an example of Point (2).

Point (1) then suggests that the same basic texture applies to  $Y_L$ : No large mixing angles should arise in  $Y_L$ . The observed  $O(1)$  atmospheric and solar mixing angles [Eqs. (1.2) and (1.4), respectively], combined with Eq. (2.8), implies that all large mixing angles must arise solely from  $W$ . While it is then tempting to suppose that the texture of  $U_{\text{MNS}}$  is determined entirely by  $W$ , there is still the matter of the small element  $U_{e3}$ , to which we next turn.

In order to distinguish contributions to  $U_{e3}$  from the charged lepton and neutrino sectors, one may adopt a separate parametrization for each of  $U_L$  and  $W$  (for which each angle is labeled with an  $E$  or  $\nu$  superscript, respectively) analogous to that of the full  $U_{\text{MNS}}$  in Eq. (2.9). Then, up to first order in the small angles  $\theta_{12}^E$ ,  $\theta_{13}^E$ ,  $\theta_{23}^E$ , and  $\theta_{13}^\nu$ , a straightforward calculation gives

$$U_{e3} = s_{13} e^{-i\delta} \simeq [\theta_{13}^\nu e^{-i\delta^\nu} - \theta_{13}^E c_{23}^\nu e^{-i\delta^E} - \theta_{12}^E s_{23}^\nu] e^{-i\beta_1^E}. \quad (3.2)$$

An analogous expression using different phase conventions appears in Ref. [22]. Regardless of the specific form, however, the important point is that  $|U_{e3}|$  is generically as large as the largest of  $\theta_{13}^\nu$ ,  $\theta_{13}^E$ , and  $\theta_{12}^E$ . The phenomenological fact from Eq. (1.5) that  $|U_{e3}| \lesssim O(0.1)$  provides additional evidence that at least two of the charged lepton mixing angles,  $\theta_{13}^E$  and  $\theta_{12}^E$ , are small. Even so, from this reasoning we see that  $U_{e3}$  may receive important contributions from  $Y_L$ . If one takes Point (1) at face value, and supposes that the reasoning leading to Eq. (3.1) applies to  $Y_L$ , then Point (3) demands that  $|U_{e3}|$  must be at least  $O(\alpha)$ , since the  $O(\alpha)$  element  $(Y_L)_{21}$  induces through the diagonalization process an  $O(\alpha)$  element  $(U_L)_{21}$ , which implies that  $\theta_{12}^E = O(\alpha)$ , and thus by Eq. (3.2),  $|U_{e3}|$  is generically at least  $O(\alpha)$ . One concludes that the current upper bound Eq. (1.5) for  $\sin^2 \theta_{13}$ , which is about  $O(\alpha^2)$ , cannot greatly exceed its actual value, for then Point (4) would be violated.



One might now argue that the assumptions of Points (1)–(3) are perhaps too restrictive; after all, perhaps there is a mild tuning of the  $O(1)$  coefficients in the Yukawa matrices that leads to a somewhat smaller  $(U_L)_{21}$  element, and hence a smaller  $\theta_{12}^E$ . Or, perhaps there exist other natural Yukawa textures in the quark sector that do not use the mechanism of Eq. (3.1). Indeed, for the remainder of this paper, let us suppose that there exist some means by which one can render the contributions of  $Y_L$  to  $U_{e3}$  at most of the order of contributions from  $W$ . In that case then, at least in terms of orders of magnitude, we are making the assumption

$$U_{\text{MNS}} \simeq W . \quad (3.3)$$

Our goal in the remainder of this paper is to show that, even in this scenario, obtaining a value for  $U_{e3} \simeq W_{13}$  as small as  $O(\alpha^2)$  is unnatural. Specifically, for a model that predicts elements of  $M_{LL}$  (or equivalently  $M_{LR}$  and  $M_{RR}$ ) to produce  $U_{e3} = O(\alpha^2)$  requires a degree of fine tuning beyond the leading  $\alpha$  orders of the matrix elements.

## B. Specific Realization

In order to see how this plays out in a full-fledged flavor model, we first introduce a bit of notation useful in discussing the problem. Let us begin by introducing a single universal small parameter  $\rho < 1$  to describe flavor symmetry breaking. In typical models, the assumed flavor group  $G_f$  with ultraviolet cutoff  $M_f$  exhibits a sequential symmetry-breaking pattern to subgroups  $G_f \supset H_1 \supset H_2 \supset \dots$ ,

$$G_f \xrightarrow{M_f \rho^{m_1}} H_1 \xrightarrow{M_f \rho^{m_2}} H_2 \xrightarrow{M_f \rho^{m_3}} \dots , \quad (3.4)$$

where the integers  $0 < m_1 < m_2 < \dots$  label sequential scales of symmetry breaking,  $M_f > M_f \rho^{m_1} > \dots$ . Models with multiple scales can then be represented by identifying the integers  $m_i$  most closely associated with the order of magnitude appropriate to each step of symmetry breaking.

Models based on  $G_f = \text{U}(2)$  [26] and its discrete subgroup  $T'$  [27, 28] exhibit two steps of symmetry breaking, with associated dimensionless parameters  $\epsilon \approx 0.04$  and  $\epsilon' \approx 0.004$ . Equivalently, one may take  $\rho \approx 0.2$  and  $m_1 = 2$ ,  $m_2 = 3$ :  $\epsilon = O(\rho^2)$  and  $\epsilon' = O(\rho^3)$ . The

charged lepton Yukawa texture for these models is

$$Y_L \simeq \begin{pmatrix} 0 & c_1 \epsilon' & 0 \\ -c_1 \epsilon' & 3c_2 \epsilon & c_3 \epsilon \\ 0 & c_4 \epsilon & c_5 \end{pmatrix}. \quad (3.5)$$

The unitary matrix  $U_L$  that diagonalizes  $Y_L Y_L^\dagger$  [Eq. (2.5)] is given by

$$U_L \approx \begin{pmatrix} 1 & \frac{c_1}{3c_2} \frac{\epsilon'}{\epsilon} & \frac{c_1 c_4}{c_5^2} \epsilon' \epsilon \\ -\frac{c_1}{3c_2} \frac{\epsilon'}{\epsilon} & 1 & \frac{c_3}{c_5} \epsilon \\ \frac{c_1 c_3}{3c_2 c_5} \epsilon' & -\frac{c_3}{c_5} \epsilon & 1 \end{pmatrix}, \quad (3.6)$$

where only the leading-order contribution is retained in each element. The charged leptonic mass texture is given by

$$Y_L^0 \approx \begin{pmatrix} \frac{c_1^2}{3c_2} \frac{\epsilon'^2}{\epsilon} & 0 & 0 \\ 0 & 3c_2 \epsilon & 0 \\ 0 & 0 & c_5 \end{pmatrix}, \quad (3.7)$$

which agrees well with the observed hierarchy  $m_e \ll m_\mu \ll m_\tau$ . However, from Eqs. (2.8), (2.9), and (3.6), the element crucial to  $U_{e3}$ , namely  $(U_L)_{21}$ , is  $O(\epsilon'/\epsilon) = O(\rho)$ . Thus,  $\rho$  and the parameter  $\alpha$  previously used are of the same order of magnitude (henceforth we use only  $\rho$ ), and the problem of a large contribution from  $Y_L$  to  $U_{e3}$  is manifested. For example, in the model of Ref. [28], which was constructed to respect bimaximal neutrino mixing, the most frequently obtained values of  $\theta_{13}$  cluster around 0.1 radians.

#### IV. CONSTRUCTING $M_{LL}$

Under the ansatz of Eq. (3.3), one can work backwards to construct the texture of  $M_{LL}$  via Eq. (2.6):

$$M_{LL} \simeq U_{\text{MNS}}^* M_{LL}^0 U_{\text{MNS}}^\dagger, \quad (4.1)$$

a calculation much in the spirit of Ref. [29]. If one assumes the seesaw mechanism Eq. (2.2), then the absolute scale of neutrino masses is determined in part by high-energy scale of  $M_{RR}$ ; in this work, however, we are interested only in ratios of neutrino masses. Using Eqs. (1.1) and (1.3), the key neutrino mass observable to accommodate is therefore

$$r \equiv \Delta m_{21}^2 / |\Delta m_{32}^2| = 0.018 - 0.047 = O(\rho^2). \quad (4.2)$$

This ratio has two well-known realizations, the normal ( $m_1 < m_2 < m_3$ ) and inverted ( $m_3 < m_1 < m_2$ ) hierarchies. One must also ask how far the lightest neutrino mass  $m_0$  lies above zero, *i.e.*, whether it vanishes as some power of  $\rho$ . Factoring out the largest neutrino mass as an overall scale, the possible normal hierarchy solutions to Eq. (4.2) are (**N1**)  $m_1 \equiv m_0 = O(\rho)$  or smaller,  $m_2 = O(\rho)$ ,  $m_3 = O(\rho^0)$ , and (**N2**)  $m_1 \equiv m_0 = O(\rho^0)$ ,  $m_2 = m_0 + O(\rho^2)$ ,  $m_3 > m_0$ , while the inverted hierarchy solutions have (**I**)  $m_1 \equiv m_0 = O(\rho^0)$ ,  $m_2 = m_0 + O(\rho^2)$ , and  $m_3 = O(\rho^0)$  or smaller. These schemes are illustrated in Fig. 1.

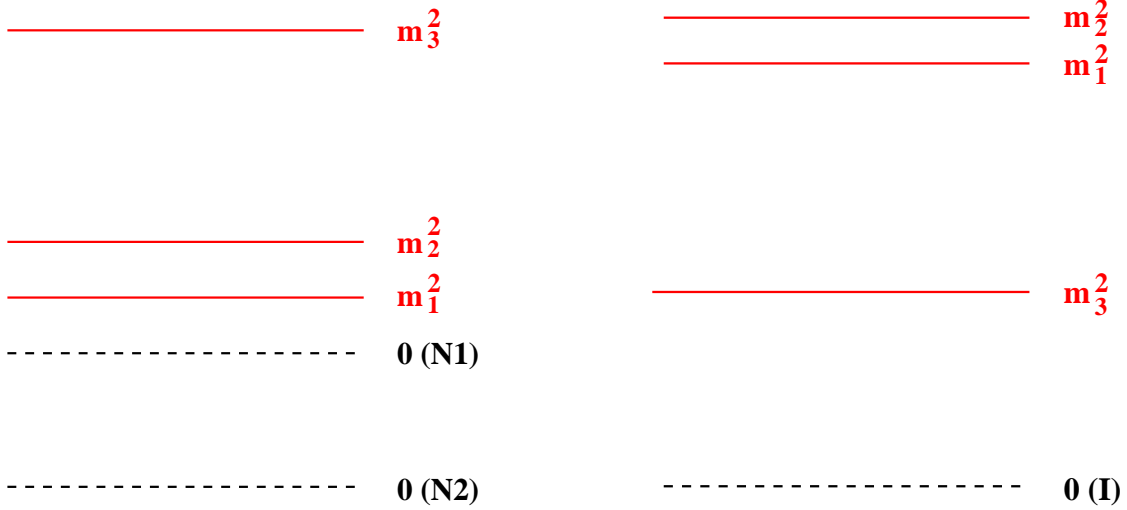


FIG. 1: The normal (**N1** and **N2**) and inverted (**I**) hierarchies. Note that **I** allows both large [ $O(\rho^0)$ ] and small [ $O(\rho^1)$ ]  $m_3$  values.

The key question then becomes this: Is it natural for a model to produce a Majorana matrix  $M_{LL}$  in any particular neutrino mass hierarchy with  $O(1)$  mixing angles  $\theta_{12}$  and  $\theta_{23}$ , but  $\theta_{13} = O(\rho^2)$ ? In this section we construct  $M_{LL}$  textures for each case.

$$\begin{aligned} \mathbf{N1} : M_{LL}^0 &= \text{diag}\{m_1^{(1)}\rho + m_1^{(2)}\rho^2, m_2^{(1)}\rho + m_2^{(2)}\rho^2, m_3^{(0)}\} \Rightarrow \\ M_{LL} &\sim \begin{pmatrix} \rho & \rho & \rho \\ \rho & 1 & 1 \\ \rho & 1 & 1 \end{pmatrix}, \end{aligned} \quad (4.3)$$

(corresponding to Type IA in Ref. [22]), while

$$\begin{aligned} \mathbf{N2} : M_{LL}^0 &= \text{diag}\{m_0, m_0 + m_2^{(2)}\rho^2, m_3\} \text{ or} \\ \mathbf{I} : M_{LL}^0 &= \text{diag}\{m_0, m_0 + m_2^{(2)}\rho^2, m_3\} \Rightarrow \end{aligned}$$

$$M_{LL} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (4.4)$$

Note that **N2** and **I** differ through the sign of  $(m_3 - m_1)$ , but otherwise give the same texture for  $M_{LL}$ . However, while the given  $M_{LL}$  textures seem simple and easily realizable [particularly the seemingly anarchic structure of Eq. (4.4)], they contain substantial internal correlations [33].

To be specific, let  $s_{13} = k\rho^2$ . Then, for **N1**, the distinct elements read

$$\begin{aligned} (M_{LL})_{11} &= (m_1^{(1)} c_{12}^2 e^{-2i\beta_1} + m_2^{(1)} s_{12}^2 e^{-2i\beta_2})\rho + (m_1^{(2)} c_{12}^2 e^{-2i\beta_1} + m_2^{(2)} s_{12}^2 e^{-2i\beta_2})\rho^2 + O(\rho^4), \\ (M_{LL})_{12} &= -s_{12}c_{12}c_{23}(m_1^{(1)} e^{-2i\beta_1} - m_2^{(1)} e^{-2i\beta_2})\rho \\ &\quad - \left[ s_{12}c_{12}c_{23}(m_1^{(2)} e^{-2i\beta_1} - m_2^{(2)} e^{-2i\beta_2}) - m_3^{(0)} s_{23}k e^{i\delta} \right] \rho^2 + O(\rho^3), \\ (M_{LL})_{13} &= s_{12}c_{12}s_{23}(m_1^{(1)} e^{-2i\beta_1} - m_2^{(1)} e^{-2i\beta_2})\rho \\ &\quad + \left[ s_{12}c_{12}s_{23}(m_1^{(2)} e^{-2i\beta_1} - m_2^{(2)} e^{-2i\beta_2}) + m_3^{(0)} c_{23}k e^{i\delta} \right] \rho^2 + O(\rho^3), \\ (M_{LL})_{22} &= s_{23}^2 m_3^{(0)} + c_{23}^2 (m_1^{(1)} s_{12}^2 e^{-2i\beta_1} + m_2^{(1)} c_{12}^2 e^{-2i\beta_2})\rho \\ &\quad + c_{23}^2 (m_1^{(2)} s_{12}^2 e^{-2i\beta_1} + m_2^{(2)} c_{12}^2 e^{-2i\beta_2})\rho^2 + O(\rho^3), \\ (M_{LL})_{23} &= s_{23}c_{23} \left[ m_3^{(0)} - (m_1^{(1)} s_{12}^2 e^{-2i\beta_1} + m_2^{(1)} c_{12}^2 e^{-2i\beta_2})\rho \right. \\ &\quad \left. - (m_1^{(2)} s_{12}^2 e^{-2i\beta_1} + m_2^{(2)} c_{12}^2 e^{-2i\beta_2})\rho^2 \right] + O(\rho^3), \\ (M_{LL})_{33} &= c_{23}^2 m_3^{(0)} + s_{23}^2 (m_1^{(1)} s_{12}^2 e^{-2i\beta_1} + m_2^{(1)} c_{12}^2 e^{-2i\beta_2})\rho \\ &\quad + s_{23}^2 (m_1^{(2)} s_{12}^2 e^{-2i\beta_1} + m_2^{(2)} c_{12}^2 e^{-2i\beta_2})\rho^2 + O(\rho^3), \end{aligned} \quad (4.5)$$

while the elements for the **N2** and **I** hierarchies read

$$\begin{aligned} (M_{LL})_{11} &= m_0(c_{12}^2 e^{-2i\beta_1} + s_{12}^2 e^{-2i\beta_2}) + m_2^{(2)} s_{12}^2 e^{-2i\beta_2} \rho^2 + O(\rho^4), \\ (M_{LL})_{12} &= -m_0 s_{12}c_{12}c_{23}(e^{-2i\beta_1} - e^{-2i\beta_2}) \\ &\quad + \left\{ m_2^{(2)} s_{12}c_{12}c_{23} e^{-2i\beta_2} - k s_{23} \left[ m_0 e^{-i\delta} (c_{12}^2 e^{-2i\beta_1} + s_{12}^2 e^{-2i\beta_2}) - m_3 e^{i\delta} \right] \right\} \rho^2 + O(\rho^4), \\ (M_{LL})_{13} &= m_0 s_{12}c_{12}s_{23}(e^{-2i\beta_1} - e^{-2i\beta_2}) \\ &\quad - \left\{ m_2^{(2)} s_{12}c_{12}s_{23} e^{-2i\beta_2} + k c_{23} \left[ m_0 e^{-i\delta} (c_{12}^2 e^{-2i\beta_1} + s_{12}^2 e^{-2i\beta_2}) - m_3 e^{i\delta} \right] \right\} \rho^2 + O(\rho^4), \\ (M_{LL})_{22} &= m_0 c_{23}^2 (s_{12}^2 e^{-2i\beta_1} + c_{12}^2 e^{-2i\beta_2}) + m_3 s_{23}^2 \\ &\quad + c_{12}c_{23} \left[ m_2^{(2)} c_{12}c_{23} e^{-2i\beta_2} + 2m_0 s_{12}s_{23}k e^{-i\delta} (e^{-2i\beta_1} - e^{-2i\beta_2}) \right] \rho^2 + O(\rho^4), \\ (M_{LL})_{23} &= -s_{23}c_{23} \left[ m_0 (s_{12}^2 e^{-2i\beta_1} + c_{12}^2 e^{-2i\beta_2}) - m_3 \right] \end{aligned}$$

$$\begin{aligned}
& -c_{12} \left[ m_2^{(2)} c_{12} s_{23} c_{23} e^{-2i\beta_2} - m_0 s_{12} k e^{-i\delta} (c_{23}^2 - s_{23}^2) (e^{-2i\beta_1} - e^{-2i\beta_2}) \right] \rho^2 + O(\rho^4), \\
(M_{LL})_{33} &= m_0 s_{23}^2 (s_{12}^2 e^{-2i\beta_1} + c_{12}^2 e^{-2i\beta_2}) + m_3 c_{23}^2 \\
& - c_{12} s_{23} \left[ 2m_0 s_{12} c_{23} k e^{-i\delta} (e^{-2i\beta_1} - e^{-2i\beta_2}) - m_2^{(2)} c_{12} s_{23} e^{-2i\beta_2} \right] \rho^2 + O(\rho^4). \quad (4.6)
\end{aligned}$$

Several comments are in order at this point. First, note the appearance of Majorana phases  $\beta_{1,2}$  in the leading terms of almost all  $M_{LL}$  elements. While it is tempting to neglect them since they remain at present unrestricted by experiment, they enter in a crucial way. In particular, models frequently produce matrices with  $\det M_{LL}^0 < 0$ , *i.e.*, some negative mass eigenvalues, indicating a relative Majorana parity difference between neutrino eigenstates. Positive eigenvalues are rescued by adding  $\pi/2$  to the appropriate relative Majorana phases  $\beta$ , as can be seen from Eqs. (2.6), (2.9), and (3.3) [34].

One popular texture [30] leading to bimaximal mixing within the normal hierarchy resembles Eq. (4.3), with the exception that  $(M_{LL})_{11} = O(\rho^2)$ . From Eqs. (4.5) we see that a very peculiar condition must occur in order for the  $O(\rho)$  coefficient of  $(M_{LL})_{11}$  to vanish:  $(m_2^{(1)}/m_1^{(1)}) \tan^2 \theta_{12} = -e^{2i(\beta_2 - \beta_1)}$ . In fact, typical textures with  $(M_{LL})_{11} = O(\rho^2)$  (*e.g.*, Ref. [28]) tend to lead to  $\Delta m_{21}^2/|\Delta m_{32}^2| = O(\rho^3)$ , which are phenomenologically disfavored. On the other hand, the leading order of each element in the texture Eq. (4.3) remains unchanged even if the mass hierarchy is fully hierarchical, *i.e.*,  $m_1 = O(\rho^2)$ , or  $m_1^{(1)} = 0$ .

The **N2** and **I** textures Eq. (4.4), on the other hand, reduce under certain circumstances to simpler forms. If  $(c_{12}^2 e^{-2i\beta_1} + s_{12}^2 e^{-2i\beta_2}) = 0$ , which is accomplished by exactly ideal solar mixing ( $\theta_{12} = \pi/4$ , which by Eq. (1.4) now appears disfavored by data; note also Ref. [31]) and Majorana parities for the corresponding mass eigenstates differing by  $|\beta_1 - \beta_2| = \pi/2$ , then  $(M_{LL})_{11} = O(\rho^2)$ . Furthermore, if in the **I** texture one additionally has  $m_3 = O(\rho)$  or smaller, then the resulting texture assumes the form (corresponding to Type IIA in Ref. [22]):

$$M_{LL} \sim \begin{pmatrix} \rho^2 & 1 & 1 \\ 1 & \rho^2 & \rho^2 \\ 1 & \rho^2 & \rho^2 \end{pmatrix}. \quad (4.7)$$

Unless  $(c_{12}^2 e^{-2i\beta_1} + s_{12}^2 e^{-2i\beta_2}) = 0$ , however, the condition  $m_3 = O(\rho)$  by itself does not change the texture of  $M_{LL}$  from the anarchic form Eq. (4.4), but it does make the lower  $2 \times 2$  block singular at leading order in  $\rho$ . Alternately, if the first two mass eigenstates have the same Majorana parity ( $\beta_1 = \beta_2$ ), then one obtains the texture (corresponding to Type IIB in

Ref. [22]):

$$M_{LL} \sim \begin{pmatrix} 1 & \rho^2 & \rho^2 \\ \rho^2 & 1 & 1 \\ \rho^2 & 1 & 1 \end{pmatrix}. \quad (4.8)$$

Finally, the small value of  $\theta_{13}$  [as indicated by the appearance of  $k$  in Eqs. (4.5) and (4.6)] leads to its near insignificance in determining  $M_{LL}$ ; it contributes to no leading-order coefficients and almost disappears from the first subleading terms as well. In fact, the primary significance of the smallness of  $\theta_{13}$  is in its absence from a large number of  $M_{LL}$  elements, for which interesting model-independent relations arise. In particular, in the normal hierarchy [Eqs. (4.5)] the lower  $2 \times 2$  block of  $M_{LL}$  is singular at leading order:  $[(M_{LL})_{22}(M_{LL})_{33} - (M_{LL})_{23}^2]_{\rho=0} = 0$ ; but the same is true for the  $O(\rho)$  coefficients of each of these elements by themselves, and similarly for the  $O(\rho^2)$  coefficients. The last of these would be spoiled if  $U_{e3} = O(\rho)$  rather than  $O(\rho^2)$ . A more significant effect of  $k$  arises in the (smaller) entries in the first row and column of  $M_{LL}$ ; for example, Eqs. (4.5) give  $(M_{LL})_{13}/(M_{LL})_{12} = -\tan\theta_{23}$ —a direct mixing observable—corrected only at relative  $O(\rho)$  by  $U_{e3}$ . If  $U_{e3} = O(\rho)$ , this relation is spoiled at leading order. The effects of  $k$  in the inverted hierarchy are more difficult to summarize in this way since they depend upon which special case [*e.g.*, Eq. (4.7) or (4.8)] of texture actually is used, but in no circumstance is  $\theta_{13}$  revealed as the sole leading-order term of any  $M_{LL}$  element.

From such considerations, we are led to the central result of this work: A value of  $\theta_{13} = O(\rho^2)$  or smaller, while certainly possible phenomenologically, is difficult to obtain through a model that predicts  $M_{LL}$  directly, since such a small  $\theta_{13}$  does not contribute significantly to any element of  $M_{LL}$ . Its effect in  $M_{LL}$  is masked by mixing with  $O(\rho^0)$ ,  $O(\rho^1)$ , and other  $O(\rho^2)$  observables, and only a model that predicts a very particular pattern of subleading effects in the elements of  $M_{LL}$  can naturally provide such a small  $\theta_{13}$ .

## V. NORMAL HIERARCHY

All of our attention thus far has focused on the detailed texture of  $M_{LL}$ . Of course, in the seesaw mechanism  $M_{LL}$  is not fundamental but derived through Eq. (2.2). The question then becomes whether it is possible to obtain textures for  $M_{LL}$  as in Eqs. (4.5) or (4.6) through particular textures of  $M_{LR}$  and  $M_{RR}$ . First, note that the seesaw formula exhibits

a symmetry: Inverting Eq. (2.2) gives

$$M_{RR} = -M_{LR}^\dagger M_{LL}^{-1} M_{LR}^* . \quad (5.1)$$

We see that this is the same as Eq. (2.2) upon exchanging  $LL \leftrightarrow RR$  and  $M_{LR} \leftrightarrow M_{LR}^T$ . Let us now consider the extension of Point (1) in Sec. III to the neutrino Yukawa mass matrix  $M_{LR}$ . In typical models, charged fermion Yukawa matrices include an  $O(1)$  33 element to represent that the Yukawa couplings leading to  $m_t$ ,  $m_b$ , and  $m_\tau$  survive in the unbroken flavor symmetry limit. This ansatz leads to phenomenologically successful small ratios for  $m_c/m_t$ ,  $m_s/m_b$ , and  $m_\mu/m_\tau$ ; in the neutrino Dirac term, however, there is no reason to anticipate such a large ratio (since neutrino masses appear not directly from the Yukawa matrix but through the seesaw mechanism). Therefore we assume a texture for  $M_{LR}$  that mirrors that of the charged fermions [compare Eq. (3.5), recalling that  $\epsilon = O(\rho^2)$  and  $\epsilon' = O(\rho^3)$ ], except that the 33 entry is no larger than the 22 or 23 entries:

$$M_{LR} \sim \rho^2 \begin{pmatrix} l_{11}\rho & l_{12}\rho & l_{13}\rho \\ l_{21}\rho & l_{22} & l_{23} \\ l_{31}\rho & l_{32} & l_{33} \end{pmatrix} . \quad (5.2)$$

We hasten to add that this ansatz for the magnitude of the 33 element is not a crucial feature of this analysis, and discuss below the effects of making the texture of  $M_{LR}$  identical to that of the charged fermions by setting  $(M_{LR})_{33} = l_{33}\rho^0$  rather than  $l_{33}\rho^2$ .

Since  $M_{LR}$  and  $M_{LR}^T$  exhibit the same texture, one expects similar textures for  $M_{LL}$  and  $M_{RR}$ . Interestingly, this is not entirely true, as we now demonstrate. Consider the determinant of both sides of Eq. (2.2); generically,  $\det M_{LR}^* = \det M_{LR}^\dagger = O(\rho^7)$ , while  $\det M_{LL} = O(\rho^2)$  from Eq. (4.3), from which follows  $\det M_{RR} = O(\rho^{12})$ . In fact, the structure of  $M_{RR}$  obtained directly by combining  $M_{LL}$  from Eq. (4.3) and  $M_{LR}$  from Eq. (5.2) yields leading-order terms

$$M_{RR} \sim \rho^3 \begin{pmatrix} a\rho^2 & bd\rho & cd\rho \\ bd\rho & b^2 + e\rho & bc + f\rho \\ cd\rho & bc + f\rho & c^2 + g\rho \end{pmatrix} , \quad (5.3)$$

where  $a, \dots, g$  are  $O(1)$  coefficients that are complicated functions of  $l_{ij}$ , the mixing angles, and the masses. Of particular note are that  $(M_{RR})_{11}/(M_{RR})_{12} = O(\rho)$ , and that the last two columns (or rows) are proportional at leading order, both in contrast to the texture of  $M_{LL}$ . The relative  $O(\rho)$  corrections provided by  $e, f, g$  lift the leading-order singularity of  $M_{RR}$ ,

providing the required  $\det M_{RR} = O(\rho^{12})$ . The overall  $\rho^3$  factor in  $M_{RR}$  merely represents our ignorance of the scale of the largest  $M_{RR}$  element; the seesaw relation Eq. (2.2) indicates that each power of  $\rho$  removed from  $M_{RR}$  must return as a power multiplying  $M_{LL}$  in Eq. (4.3).

It is important to reiterate that Eq. (5.2), although motivated by reasonable criteria, remains an ansatz. If  $M_{LR}$  possesses a number of elements smaller than suggested above (texture zeroes) or a correlation among its leading  $l_{ij}$  coefficients, owing to the constraints of a flavor symmetry, then an  $M_{RR}$  structure substantially different from Eq. (5.3) may be mandated in order to obtain the desired  $M_{LL}$ .

In order to achieve the desired form of  $M_{LL}$  with a generic  $M_{LR}$ , a great deal of structure must be incorporated into  $M_{RR}$ . We demonstrate this by using *generic*  $M_{LR}$  and  $M_{RR}$  textures of the forms given by Eqs. (5.2) and (5.3) to compute  $M_{LL}$ . The result of this exercise does indeed give a texture of the form Eq. (4.3), for which  $[(M_{LL})_{22}(M_{LL})_{33} - (M_{LL})_{23}^2]_{\rho=0} = 0$  [but not for the coefficients of the  $O(\rho)$  terms, as holds for Eqs. (4.5)]. It also gives mass eigenvalues in the normal hierarchy,  $m_3 = O(\rho^0)$ ,  $m_{1,2} = O(\rho)$  and  $\theta_{23} = O(\rho^0)$ . However, the value of  $\theta_{13}$  obtained from these textures is  $O(\rho)$ . In order to obtain  $\theta_{13} = O(\rho^2)$ , it would be necessary to modify Eq. (5.3) by the inclusion of highly correlated subleading terms, which appears to violate the spirit of naturalness and aversion to fine tuning in model building.

A few words are in order regarding the derivation of neutrino observables mentioned in the previous paragraph. It is not actually necessary to solve the full eigenvector problem to determine the leading contribution to most observables. Rather, one begins by reversing the argument above Eq. (5.3), to find first that  $\det M_{LL} = O(\rho^2)$ . Next, since  $M_{LL}$  is approximately diagonalized by unitary  $U_{\text{MNS}}$  via Eq. (4.1), then

$$\text{Tr} M_{LL} \simeq \text{Tr}(U_{\text{MNS}}^\dagger U_{\text{MNS}}^* M_{LL}^0) . \quad (5.4)$$

Inasmuch as effects of the complex phase  $\delta$  in  $U_{\text{MNS}}$  are suppressed by the smallness of  $s_{13}$ ,  $U_{\text{MNS}}^\dagger U_{\text{MNS}}^*$  is just the phase matrix  $\text{diag}(e^{-2i\beta_1}, e^{-2i\beta_2}, 1)$ , meaning that  $\text{Tr} M_{LL}$  is essentially the sum of the eigenvalues, and  $\text{Tr} M_{LL}^2$  is essentially the sum of their squares.  $\det M_{LL}$ , likewise, is the product of the eigenvalues up to a phase. These phases are inessential to our discussion and hence are neglected. Since in the present case  $\text{Tr} M_{LL} = \mu + O(\rho)$ , and the  $O(\rho^0)$  term of  $\text{Tr} M_{LL}^2$  turns out to be  $\mu^2$ , one concludes that precisely one eigenvalue is  $O(\rho^0)$ :  $\mu = m_3^{(0)} \Rightarrow m_3 = m_3^{(0)} + O(\rho)$ , while  $m_1$  and  $m_2$  are each  $O(\rho^1)$ . The  $O(\rho)$  coefficient  $m_3^{(1)}$  can then be determined since the  $O(\rho)$  coefficient of  $\text{Tr} M_{LL}^2$  is just  $2m_3^{(0)}m_3^{(1)}$ .



The leading-order terms of  $m_3$  thus obtained may then be fed into the equation  $(M_{LL} - m_3 \mathbb{1})u_3 = 0$  to determine the corresponding eigenvector  $u_3$ , which once normalized is the third column of  $U_{\text{MNS}}$ . From this procedure one obtains  $(u_3)_2/(u_3)_3 = O(\rho^0)$  and  $(u_3)_1/(u_3)_3 = O(\rho)$ . Normalization of  $u_3$  and a glance at Eq. (2.9) then demonstrates that  $\theta_{23} = O(\rho^0)$  and  $\theta_{13} = O(\rho)$ .

If, instead of the form Eq. (5.2), one insists on a texture for  $M_{LR}$  completely parallel to that of the charged fermions [*i.e.*,  $(M_{LR})_{33} = l_{33}\rho^0$ ], then the powers of  $\rho$  in the texture for  $M_{RR}$  change compared to Eq. (5.3), but its rank and the basic predictions for neutrino observables do not. Specifically, one finds

$$M_{RR} \sim \frac{1}{\rho} \begin{pmatrix} a\rho^6 & bd\rho^5 & cd\rho^3 \\ bd\rho^5 & (b^2 + e\rho)\rho^4 & (bc + f\rho)\rho^4 \\ cd\rho^3 & (bc + f\rho)\rho^2 & c^2 + g\rho \end{pmatrix}, \quad (5.5)$$

from which one still obtains  $\theta_{23} = O(\rho^0)$  and  $\theta_{13} = O(\rho)$ . The overall  $1/\rho$  indicates not a singularity in the flavor limit, but again reveals the fact that the inverse seesaw Eq. (5.1) allows the portability of overall  $\rho$  powers among  $M_{LR}$ ,  $M_{LL}$ , and  $M_{RR}$ .

## VI. INVERTED HIERARCHY

For completeness, let us follow the same logic of using the inverted seesaw mechanism [Eq. (5.1)] for the **N2** and **I** hierarchies, using the same ansatz [Eq. (5.2)] as above for  $M_{LR}$ . Starting with the explicit forms for the elements of  $M_{LL}$  given in Eqs. (4.6) (but with no further constraints), one obtains the explicit form

$$M_{RR} \sim \rho^4 \begin{pmatrix} \rho^2 & \rho & \rho \\ \rho & 1 & 1 \\ \rho & 1 & 1 \end{pmatrix}, \quad (6.1)$$

where comments from the previous section on the portability of the overall factor of  $\rho^4$  between  $M_{LL}$  and  $M_{RR}$  still apply. One finds  $\det M_{RR} = O(\rho^{14})$ , the value obtained through naive power counting on Eq. (6.1), as well as the multiplicative nature of the determinant [ $\det M_{LL} = O(\rho^0)$ ,  $\det M_{LR} = O(\rho^7)$ ] using Eq. (5.1). The lower  $2 \times 2$  block is singular, it turns out, only if  $(c_{12}^2 e^{-2i\beta_1} + s_{12}^2 e^{-2i\beta_2}) = 0$  [which leads to  $M_{LL}$  as given in Eq. (4.7), but again is disfavored by data], or if  $l_{22}l_{33} - l_{23}l_{32} = 0$ , *i.e.*, the lower  $2 \times 2$  block of  $M_{LR}$  (which gives

the leading-order contribution to  $\det M_{LR}$ ) is singular. In particular, the condition  $\beta_1 = \beta_2$  by itself, which gives rise to the texture Eq. (4.8), still gives a texture of the form Eq. (6.1).

If  $m_3 = O(\rho)$ , then one obtains the form [cf. Eq. (5.3)]

$$M_{RR} \sim \rho^3 \begin{pmatrix} d^2 \rho^2 & bd\rho & cd\rho \\ bd\rho & b^2 & bc \\ cd\rho & bc & c^2 \end{pmatrix}, \quad (6.2)$$

for which one finds  $\det M_{RR} = O(\rho^{13})$  using Eq. (5.1), owing to the extra power of  $\rho$  in  $\det M_{LL}$  from  $m_3$ . The naive power-counting result from Eq. (6.2) is  $\det M_{RR} = O(\rho^{11})$ , but this difference is resolved by the fact (apparent from this form) that  $M_{RR}$  is only rank 1.

If one changes Eq. (5.2) to make  $(M_{LR})_{33} = l_{33}\rho^0$  in parallel to the charged fermion textures, then again the specific powers emerging in the derived  $M_{RR}$  change compared to Eq. (6.2), but its rank and the size of the predicted observables do not. Specifically,

$$M_{RR} \sim \frac{1}{\rho} \begin{pmatrix} d^2 \rho^6 & bd\rho^5 & cd\rho^3 \\ bd\rho^5 & b^2 \rho^4 & bc\rho^2 \\ cd\rho^3 & bc\rho^2 & c^2 \end{pmatrix}. \quad (6.3)$$

In each case, however, it is clear that the assumed value  $s_{13} = O(\rho^2)$  is buried even more deeply in the structure of  $M_{RR}$  than in the normal hierarchy. This is apparent already from Eqs. (4.6), in which  $k$  appears only in combination with the second-order mass difference  $m_2^{(2)}$ . The problem traces back to larger  $[O(\rho^0)]$  values of the “upper” masses  $m_{1,2}$  in the inverted hierarchy, which tend to make the elements even in the first row and column of  $M_{LL}$  larger, thus obscuring the few places where  $s_{13}$  might hope to dominate. Indeed, a calculation analogous to that in the previous section shows that the mixing angle  $s_{13}$  obtained from either Eq. (6.2) or (6.3), for example, is generically  $O(\rho^0)$ .

## VII. CONCLUSIONS

The low-energy neutrino observables, as determined from the latest data, include two  $O(1)$  mixing angles and a mass hierarchy ratio which, in terms of the universal flavor symmetry-breaking parameter  $\rho = O(0.1)$ , is  $O(\rho^2)$  (*i.e.*, not excessively small). This, we have seen, is sufficient to obscure a truly small  $[O(\rho^2)]$  value for  $|U_{e3}|$  emerging from a model that predicts individual elements of the Majorana mass matrix  $M_{LL}$ . In other words, a certain amount of

correlation between the elements of  $M_{LL}$ , extending down to coefficients subleading in  $\rho$ , is necessary in order to guarantee a generically small  $|U_{e3}|$ . Such correlations may be viewed as unappealing, a form of fine tuning.

We have illustrated the consequences of this scenario by computing the explicit texture of  $M_{LL}$  in both normal [Eqs. (4.5)] and inverted [Eqs. (4.6)] hierarchies, using generic  $O(1)$  values for solar and atmospheric angles and  $O(\rho^2)$  for the neutrino mass splittings ratio, and assuming a value of  $|U_{e3}| = O(\rho^2)$ . Although the most natural picture of flavor breaking predicts an  $O(\rho)$  contribution to  $U_{e3}$  arising from diagonalization of the charged lepton Yukawa matrix  $Y_L$ , we first supposed that such a contribution is somehow suppressed. We then adopted a form for the neutrino Dirac mass matrix  $M_{LR}$  [Eq. (5.2)] similar to those typically used for the charged fermions. Next, we employed the simple seesaw mechanism to derive textures for the right-handed Majorana mass matrix  $M_{RR}$ , and asked whether a texture of the *generic* form thus derived—without excessive correlation of its subleading terms—naturally gives rise to observables at the desired orders of magnitude. In all cases, we found that  $O(\rho^2)$  values of  $|U_{e3}|$  do not arise naturally, in the sense of the criteria laid out in Sec. III.

It is important to consider whether these criteria, or the form of the Dirac matrix used in Eq. (5.2), is too restrictive. In fact, it should be possible to evade these constraints if certain key elements in  $M_{LR}$  should turn out to be correlated at leading order in  $\rho$ , or smaller than suggested by Eq. (5.2). An appropriately restrictive flavor symmetry, for example, can provide such useful correlations or texture zeroes. One must nevertheless keep in mind that these matrix elements are defined at high energy scales at which the flavor symmetry breaks, and may mix through the evolution of renormalization group equations down to low scales, thus potentially obscuring such constraints.

Finally, we point out that the expressions derived here, with minor modifications, can be used in another way. Even though the purpose of this paper has been the negative result of showing that  $O(\rho^2)$  values of  $|U_{e3}|$  are difficult to achieve, the results of Eqs. (4.5)–(4.6) are still useful in the likely physical situation, *i.e.*,  $|U_{e3}| = O(\rho)$ , if one merely replaces each occurrence of  $U_{e3} = k\rho^2 e^{-i\delta}$  by  $k\rho e^{-i\delta}$ .

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- [33] Possible subleading contributions to  $m_3$  in Eq. (4.3) are removed by redefining the overall (undetermined) scale of the matrix  $M_{LL}$ . In the case of Eq. (4.4) this cannot be done simultaneously for  $m_0$  and  $m_3$ , but subleading corrections to  $m_3$ , if desired, may be inserted into the following expressions by hand.
- [34] In contrast, similar sign differences between eigenvalues of the Yukawa matrices are removed by redefining Dirac fermion fields via  $\psi \rightarrow \gamma_5 \psi$ .